Oscillation Criteria of First Order Linear Difference Equations with Delay Argument

G. E. Chatzarakis,[a] R. Koplatadze,[b] and I. P. Stavroulakis[a]

Abstract

This paper presents new sufficient conditions for the oscillation of all proper solutions of the first order linear difference equation with delay argument

$$\Delta u(k) + p(k)\, u(\tau(k)) = 0, \quad k \in N,$$

where $\Delta u(k) = u(k+1) - u(k)$, $p : N \to R_+$, $\tau : N \to N$ and $\lim\limits_{k \to +\infty} \tau(k) = +\infty$. Examples illustrating the results are given. It is to be pointed out that this is the first paper dealing with the oscillatory behaviour of the equation in the case of a general delay argument $\tau(k)$.

[a] Department of Mathematics, University of Ioannina, 451 10, Greece, ipstav@cc.uoi.gr
[b] A. Razmadze Mathematical Institute Georgia Academy of Sciences, 1, M. Aleksidze st., Tbilisi, Georgia

1 Introduction

Consider the first order linear difference equation with delay argument

$$\Delta u(k) + p(k)\, u(\tau(k)) = 0, \quad k \in N, \tag{E}$$

where $\Delta u(k) = u(k+1) - u(k)$, $p : N \to R_+$, $\tau : N \to N$ and $\lim\limits_{k \to +\infty} \tau(k) = +\infty$.

Strong interest in equation (E) is motivated by the fact, that it represents a discrete analogue of the delay differential equation (see [12] and the references cited therein)

$$x'(t) + p(t)\,x(\tau(t)) = 0, \quad p(t) \geq 0, \quad \tau(t) \leq t \quad \text{for} \quad t \geq 0.$$

By a *proper solution* of Eq.(E) we mean a function $u : N_{n_0} \longrightarrow R$, $n_0 = \min\{\tau(k) : k \in N_n\}$, $N_n = \{n, n+1, \dots\}$ which satisfies Eq.(E) on N_n and $\sup\{|u(i)| : i \geq k\} > 0$ for $k \in N_{n_0}$.

A proper solution $u : N \longrightarrow R$ of Eq.(E) is said to be *oscillatory* (around zero) if for every positive integer n there exist $n_1, n_2 \in N_n$, such that $u(n_1)u(n_2) \leq 0$. Otherwise, the solution is said to be *non-oscillatory*. In other words, a proper solution u is oscillatory if it is neither eventually positive nor eventually negative.

In the last few decades the oscillation theory of delay differential equations has been extensively developed. The oscillation theory of discrete analogues of delay differential equations has also attracted growing attention in the recent few years. In particular, the problem of establishing sufficient conditions for the oscillation of all solutions of the equation

$$\Delta u(k) + p(k)\,u(k-n) = 0, \quad k \in N \tag{E$_1$}$$

has been the subject of many recent investigations. See for example [2–11, 13–16] and the references cited therein.

In 1989, Erbe and Zhang [6], proved that, if $p(k) \geq 0$, then either one of the following conditions

$$\liminf_{k \to +\infty} p(k) > \frac{n^n}{(n+1)^{n+1}} \tag{1.1}$$

or

$$\limsup_{k \to +\infty} \sum_{i=k-n}^{k} p(i) > 1 \tag{1.2}$$

implies that all solutions of Eq.(E_1) oscillate.

In the same year, Ladas, Philos and Sficas [9], proved that the same conclusion holds if $p(k) \geq 0$ and

$$\liminf_{k \to +\infty} \left(\frac{1}{n} \sum_{i=k-n}^{k-1} p(i) \right) > \frac{n^n}{(n+1)^{n+1}}\,. \tag{1.3}$$

It is interesting to establish sufficient conditions for the oscillation of all solutions of Eq.(E_1) when the conditions (1.2) and (1.3) are not satisfied. Many researchers focused on the improvement of the upper bound of the ratio $u(k-n)/u(k)$ for possible non-oscillatory solutions u of Eq.(E_1). In 1993, Yu, Zhang and Qian [16], and Lalli and Zhang [10], trying to improve (1.2) established some false oscillation conditions due to the fact that both were based on an erroneous discrete version of the Koplatadze-Chanturia lemma [8]. For more details the reader is referred to [5,3].

In 1995, Stavroulakis [14], proved that if

$$0 < \alpha := \liminf_{k\to+\infty} \sum_{i=k-n}^{k-1} p(i) \le \left(\frac{n}{n+1}\right)^{n+1} \quad \text{and} \quad \limsup_{k\to+\infty} p(k) > 1 - \frac{\alpha^2}{4}$$

then all solutions of Eq.(E_1) oscillate.

In 1999, Domshlak [5], and in 2000, Cheng and Zhang [3], established the following lemmas respectively, which may be looked upon as discrete versions of Koplatadze-Chanturia lemma [8].

Lemma 1.1 *([5]) Assume that u is an eventually positive solution of Eq.(E_1) and that*

$$\sum_{i=k-n}^{k-1} p(i) \ge \alpha > 0 \quad for\ large\ \ k.$$

Then

$$u(k) > \frac{\alpha^2}{4}\, u(k-n) \quad for\ large\ \ k. \tag{1.4}$$

Lemma 1.2 *([3]) Assume that u is an eventually positive solution of Eq.(E_1) and that*

$$\sum_{i=k-n}^{k-1} p(i) \ge \alpha > 0 \quad for\ \ large\ \ k.$$

Then

$$u(k) > \alpha^n\, u(k-n) \quad for\ large\ \ k. \tag{1.5}$$

In 2004, Stavroulakis [15], based on the above two lemmas, established the following theorem.

Theorem 1.1 *([15]) Assume that*

$$0 < \alpha := \liminf_{k\to+\infty} \sum_{i=k-n}^{k-1} p(i) \le \left(\frac{n}{n+1}\right)^{n+1}.$$

Then either one of the conditions

$$\limsup_{k\to+\infty} \sum_{i=k-n}^{k-1} p(i) > 1 - \frac{\alpha^2}{4} \tag{1.6}$$

or

$$\limsup_{k\to+\infty} \sum_{i=k-n}^{k-1} p(i) > 1 - \alpha^n \tag{1.7}$$

implies that all solutions of Eq.(E_1) *oscillate.*

In 2006, Chatzarakis and Stavroulakis [2], established the following lemma.

Lemma 1.3 *([2]) Assume that u is an eventually positive solution of Eq.*(E_1) *and that*

$$\sum_{i=k-n}^{k-1} p(i) \ge \alpha > 0 \quad \textit{for large} \quad k.$$

Then

$$u(k) > \frac{\alpha^2}{2(2-\alpha)} u(k-n) \quad \textit{for large} \quad k. \tag{1.8}$$

Based on the above lemma, they established the following theorem.

Theorem 1.2 *([2]) Assume that*

$$0 < \alpha := \liminf_{k\to+\infty} \sum_{i=k-n}^{k-1} p(i) \le \left(\frac{n}{n+1}\right)^{n+1}$$

and

$$\limsup_{k\to+\infty} \sum_{i=k-n}^{k-1} p(i) > 1 - \frac{\alpha^2}{2(2-\alpha)}. \tag{1.9}$$

Then all solutions of Eq. (E_1) *oscillate.*

In this paper, the authors improve the upper bound of the ratio $u(\tau(k))/u(k+1)$ for possible non-oscillatory proper solutions u of Eq.(E) and derive new sufficient oscillation conditions. It is to be emphasized that this is the first paper dealing with the oscillatory behaviour of Eq.(E) in the case of a general delay argument $\tau(k)$.

2 Oscillation Criteria for Eq. (E)

In this section we first establish two lemmas which will be used in the proof of our main results.

Consider the difference inequality

$$\Delta u(k) + q(k)\, u(\sigma(k)) \le 0, \quad k \in N, \tag{2.1}$$

where

$$q : N \to R_+, \quad \sigma : N \to N \quad \text{and} \quad \lim_{k\to+\infty} \sigma(k) = +\infty. \tag{2.2}$$

Lemma 2.1 *Let*

$$\liminf_{k\to+\infty} \sum_{i=\tau(k)}^{k-1} p(i) = \alpha > 0, \tag{2.3}$$

$$\sigma(k) \le \tau(k) \le k-1, \quad p(k) \le q(k) \quad \textit{for} \quad k \in N \tag{2.4}$$

and $u : N_{n_0} \to (0,+\infty)$ *be a positive proper solution of* (2.1) *for a certain* $n_0 \in N$. *Then Eq.*(E) *has a proper solution* $u_* : N_{n_1} \to (0,+\infty)$ *such that*

$$0 < u_*(k) \le u(k) \quad \textit{for} \quad k \in N_{n_1}, \tag{2.5}$$

where $n_1 > n_0$ *is a sufficiently large natural number.*

Proof. Let $u : N_{n_0} \to (0,+\infty)$ be a positive proper solution of (2.1). By (2.2) and (2.3), it is clear that there exists $n_1 \in N_{n_0}$ such that

$$u(\sigma(k)) > 0 \quad \text{and} \quad \sum_{i=\tau(k)}^{k-1} p(i) > 0 \quad \text{for} \quad k \in N_{n_1}. \tag{2.6}$$

From (2.1), we have

$$u(k) \ge \sum_{i=k}^{+\infty} q(i)\, u(\sigma(i)) \quad \text{for} \quad k \in N_{n_1}. \tag{2.7}$$

Assume that $n_* = \min\{\tau(k) : k \in N_{n_1}\}$ and consider the sequence of functions $u_i : N_{n_*} \to R$ $(i = 1, 2, \dots\}$ defined as follows

$$u_1(k) = u(k) \quad \text{for} \quad k \in N_{n_*},$$

$$u_j(k) = \begin{cases} \sum_{i=k}^{+\infty} p(i)\, u_{j-1}(\tau(i)) & \text{for} \quad k \in N_{n_1}, \\ u(k) & \text{for} \quad k \in [n_*, n_1) \quad (j = 2, 3, \dots). \end{cases} \tag{2.8}$$

By (2.4), (2.7) and using the fact that the function u is nonincreasing, we have

$$u_2(k) = \sum_{i=k}^{+\infty} p(i)\, u_1(\tau(i)) \le \sum_{i=k}^{+\infty} q(i)\, u(\sigma(i)) \le u(k) = u_1(k) \quad \text{for} \quad k \ge n_1.$$

Thus

$$u_j(k) \le u_{j-1}(k) \quad \text{for} \quad k \in N_{n_1} \quad (j = 2, 3, \dots). \tag{2.9}$$

Denote $\lim_{j\to+\infty} u_j(k) = u_*(k)$ (according to (2.9) this limit exists). Therefore, from (2.8), we get

$$u_*(k) = \sum_{i=k}^{+\infty} p(i)\, u_*(\tau(i)) \quad \text{for} \quad k \in N_{n_1}. \tag{2.10}$$

Now, we will show that $u_*(k) > 0$ for $k \ge n_1$. Assume, for the sake of contradiction, that there exists $n_2 \ge n_1$ such that $u_*(k) = 0$ for $k \in N_{n_2}$ and $u_*(k) > 0$ for $k \in [n_*, n_2)$. Denote by N^* the set of natural numbers n for which $\tau(k) = n_2$ and $n^* = \min N^*$. By (2.10) and (2.4) it is clear that $n^* \ge n_2$. Therefore, if $c = \min\{u_*(\tau(i)) : \tau(n^*) \le i \le n^* - 1\} > 0$, by (2.4) and (2.6), we have

$$u_*(n_2) = \sum_{i=n_2}^{+\infty} p(i)\, u_*(\tau(i)) \ge \sum_{i=\tau(n^*)}^{n^*-1} p(i)\, u_*(\tau(i)) \ge c \sum_{i=\tau(n^*)}^{n^*-1} p(i) > 0,$$

which, in view of $u_*(n_2) = 0$, leads to a contradiction. Therefore, $u_*(k) > 0$ for $k \ge n_1$.

Hence Eq.(E) has a proper solution u_* satisfying $0 < u_*(k) \le u(k)$ for $k \in N_{n_1}$. The proof is complete.

Lemma 2.2 *Assume that u is a positive proper solution of Eq.(E), where*

$$\begin{gathered} p : N \to R_+, \quad \tau : N \to N \quad \textit{is nondecreasing function,} \\ \tau(k) \le k - 1, \quad \textit{for} \quad k \in N, \quad \lim_{k\to+\infty} \tau(k) = +\infty \end{gathered} \tag{2.11}$$

and

$$\liminf_{k\to+\infty} \sum_{i=\tau(k)}^{k-1} p(i) = \alpha \in (0, 1]. \tag{2.12}$$

Then

$$\limsup_{k\to+\infty}\frac{u(\tau(k))}{u(k+1)}\le\left(\frac{1+\sqrt{1-\alpha}}{\alpha}\right)^2. \tag{2.13}$$

If, additionally, $p(k)\ge 1-\sqrt{1-\alpha}$ *for large* k*, then*

$$\limsup_{k\to+\infty}\frac{u(\tau(k))}{u(k+1)}\le\frac{1-\alpha+\sqrt{1-\alpha}}{\alpha^2}. \tag{2.14}$$

Proof. By (2.12), it is clear that, for any $\varepsilon\in(0,\alpha)$ there exists $n_0=n_0(\varepsilon)\in N$ such that

$$\sum_{i=\tau(k)}^{k-1}p(i)\ge\alpha-\varepsilon\quad\text{for}\quad k\in N_{n_0}. \tag{2.15}$$

Since u is a positive proper solution of Eq.(E), then there exists $n_1\in N_{n_0}$ such that

$$u(\tau(k))>0\quad\text{for}\quad k\in N_{n_1}.$$

Thus, from Eq.(E), we have

$$u(k+1)-u(k)=-p(k)\,u(\tau(k))\le 0$$

and so u is an eventually nonincreasing function of positive numbers.

From (2.15), it is clear that, if $\omega\in(0,\alpha-\varepsilon)$, there exists $k^*\ge k$ such that

$$\sum_{i=k}^{k^*-1}p(i)<\omega\quad\text{and}\quad\sum_{i=k}^{k^*}p(i)\ge\omega. \tag{2.16}$$

This is because in the case where $p(k)<\omega$, it is clear that, there exists $k^*>k$ such that (2.16) is satisfied, while in the case where $p(k)\ge\omega$, then $k^*=k$, and therefore

$$\sum_{i=k}^{k^*-1}p(i)=\sum_{i=k}^{k-1}p(i)\ \text{(by which we mean)}=0<\omega\quad\text{and}\quad\sum_{i=k}^{k^*}p(i)=\sum_{i=k}^{k}p(i)=p(k)\ge\omega.$$

That is, in both cases (2.16) is satisfied. Thus

$$\sum_{i=\tau(k^*)}^{k-1}p(i)=\sum_{i=\tau(k^*)}^{k^*-1}p(i)-\sum_{i=k}^{k^*-1}p(i)\ge(\alpha-\varepsilon)-\omega.$$

Now, summing up Eq.(E) first from k to k^* and then from $\tau(k^*)$ to $k-1$, and using the fact that the function u is nonincreasing and the function τ is nondecreasing, we have

$$u(k) - u(k^*+1) = \sum_{i=k}^{k^*} p(i)\, u(\tau(i)) \geq \left(\sum_{i=k}^{k^*} p(i)\right) u(\tau(k^*)) \geq \omega\; u(\tau(k^*))$$

or

$$u(k) \geq u(k^*+1) + \omega u(\tau(k^*)) \tag{2.17}$$

and then

$$u(\tau(k^*)) - u(k) = \sum_{i=\tau(k^*)}^{k-1} p(i)\, u(\tau(i)) \geq \left(\sum_{i=\tau(k^*)}^{k-1} p(i)\right) u(\tau(k-1)) \geq [(\alpha-\varepsilon)-\omega]u(\tau(k-1))$$

or

$$u(\tau(k^*)) \geq u(k) + [(\alpha-\varepsilon)-\omega]u(\tau(k-1)). \tag{2.18}$$

Combining inequalities (2.17) and (2.18), we obtain

$$u(k) \geq u(k^*+1) + \omega[u(k) + ((\alpha-\varepsilon)-\omega)u(\tau(k-1))]$$

or

$$u(k) \geq \frac{\omega[(\alpha-\varepsilon)-\omega]}{1-\omega} u(\tau(k-1)). \tag{2.19}$$

Observe that the function $f : (0,\alpha) \to (0,1)$ defined as

$$f(\omega) :\;\; = \frac{\omega[(\alpha-\varepsilon)-\omega]}{1-\omega} \tag{2.20}$$

attains its maximum at $\omega = 1-\sqrt{1-(\alpha-\varepsilon)}$, which is equal to

$$f_{\max} = \left(1-\sqrt{1-(\alpha-\varepsilon)}\right)^2.$$

Thus, for $\omega = 1-\sqrt{1-(\alpha-\varepsilon)} \in (0, \alpha-\varepsilon)$ inequality (2.19) becomes

$$u(k) \geq \left(1-\sqrt{1-(\alpha-\varepsilon)}\right)^2 u(\tau(k-1))$$

or

$$\frac{u(\tau(k-1))}{u(k)} \leq \left(\frac{1+\sqrt{1-(\alpha-\varepsilon)}}{\alpha-\varepsilon}\right)^2 \tag{2.21}$$

and, for large k, we have

$$\frac{u(\tau(k))}{u(k+1)} \le \left(\frac{1+\sqrt{1-(\alpha-\varepsilon)}}{\alpha-\varepsilon} \right)^2 .$$

Hence,

$$\limsup_{k\to+\infty} \frac{u(\tau(k))}{u(k+1)} \le \left(\frac{1+\sqrt{1-(\alpha-\varepsilon)}}{\alpha-\varepsilon} \right)^2 ,$$

which, for arbitrarily small values of ε, implies (2.13).

Next we consider the particular case where $p(k) \ge 1-\sqrt{1-\alpha}$.

In this case, from Eq.(E) we have

$$u(k) = u(k+1) + p(k)\, u(\tau(k)) \ge (1-\sqrt{1-\alpha})\, u(\tau(k)). \tag{2.22}$$

Now, summing up Eq.(E) from $\tau(k)$ to $k-1$, and using the fact that the function u is nonincreasing and the function τ is nondecreasing, we have

$$u(\tau(k)) - u(k) = \sum_{i=\tau(k)}^{k-1} p(i)\, u(\tau(i)) \ge \left(\sum_{i=\tau(k)}^{k-1} p(i) \right) u(\tau(k-1)) \ge (\alpha-\varepsilon) u(\tau(k-1))$$

or

$$u(\tau(k)) \ge u(k) + (\alpha-\varepsilon)\; u(\tau(k-1)). \tag{2.23}$$

Combining inequalities (2.22) and (2.23), we obtain

$$u(k) \ge (1-\sqrt{1-\alpha})[u(k) + (\alpha-\varepsilon) u(\tau(k-1))]$$

or

$$\frac{u(\tau(k-1))}{u(k)} \le \frac{1-\alpha+\sqrt{1-\alpha}}{\alpha(\alpha-\varepsilon)} \tag{2.24}$$

and, for large k,

$$\frac{u(\tau(k))}{u(k+1)} \le \frac{1-\alpha+\sqrt{1-\alpha}}{\alpha(\alpha-\varepsilon)}.$$

Hence

$$\limsup_{k\to+\infty} \frac{u(\tau(k))}{u(k+1)} \le \frac{1-\alpha+\sqrt{1-\alpha}}{\alpha(\alpha-\varepsilon)}.$$

The last inequality, for arbitrarily small values of ε, implies (2.14). The proof is complete.

Theorem 2.1 *Assume that* $\tau(k) \leq k$ *and*

$$\sigma(k) = \max\{\tau(s) : 1 \leq s \leq k, \quad s \in N\}. \tag{2.25}$$

If

$$\limsup_{k \to +\infty} \sum_{i=\sigma(k)}^{k} p(i) > 1, \tag{2.26}$$

then all proper solutions of Eq.(E) *oscillate.*

Proof. Assume, for the sake of contradiction, that $u_0 : N_{n_0} \to (0, +\infty)$ is a positive proper solution of Eq.(E).

Since the function u_0 is nonincreasing and $\sigma(k) = \max\{\tau(s) : 1 \leq s \leq k, \ s \in N\}$ then, for sufficiently large $k \in N_{n_0}$, u_0 satisfies the following inequality

$$\Delta u_0(k) + p(k)\, u_0(\sigma(k)) \leq 0.$$

Summing up the last inequality from $\sigma(k)$ to k, and using the fact that the function u_0 is nonincreasing and the function σ is nondecreasing, we have

$$u_0(\sigma(k))\left(\sum_{i=\sigma(k)}^{k} p(i) - 1\right) \leq 0.$$

Therefore, for sufficiently large k

$$\sum_{i=\sigma(k)}^{k} p(i) \leq 1,$$

which contradicts (2.26). The proof is complete.

Remark 2.1 *In the special case of Eq.*(E_1) *the above condition* (2.26) *leads to the condition* (1.2) *presented in [6].*

Theorem 2.2 *Assume that*

$$\liminf_{k \to +\infty} \sum_{i=\tau(k)}^{k-1} p(i) = \alpha \in (0, 1] \tag{2.12}$$

and

$$\limsup_{k\to+\infty}\sum_{i=\sigma(k)}^{k} p(i) > 1-\left(1-\sqrt{1-\alpha}\right)^2, \tag{2.27}$$

where

$$\sigma(k)=\max\left\{\tau(s): 1\le s\le k,\quad s\in N\right\}. \tag{2.25}$$

Then all proper solutions of Eq.(E) *oscillate.*

If, additionally, $p(k)\ge 1-\sqrt{1-\alpha}$ *for large* k*, and*

$$\limsup_{k\to+\infty}\sum_{i=\sigma(k)}^{k} p(i) > 1-\alpha\frac{1-\sqrt{1-\alpha}}{\sqrt{1-\alpha}} \tag{2.28}$$

then all proper solutions of Eq.(E) *oscillate.*

Proof. We will first show that

$$\liminf_{k\to+\infty}\sum_{i=\sigma(k)}^{k-1} p(i)=\alpha. \tag{2.29}$$

Indeed, since $\tau(k)\le\sigma(k)$, then by (2.12), it is obvious that

$$\liminf_{k\to+\infty}\sum_{i=\sigma(k)}^{k-1} p(i)\le\liminf_{k\to+\infty}\sum_{i=\tau(k)}^{k-1} p(i)=\alpha. \tag{2.30}$$

Thus, there exists a subsequence $\{k_i\}_{i=1}^{+\infty}$ of natural numbers such that $k_i\uparrow +\infty$ for $i\to+\infty$ and

$$\liminf_{k\to+\infty}\sum_{i=\sigma(k)}^{k-1} p(i)=\lim_{k\to+\infty}\sum_{j=\sigma(k_i)}^{k_i-1} p(j). \tag{2.31}$$

On the other hand, from the definition of the function σ and taking into account that $\lim_{k\to+\infty}\tau(k)=+\infty$ for any k_i $(i=1,2,\dots)$ there exists $k_i'\le k_i$ such that $\sigma(k)=\sigma(k_i)$ since $k_i'\le k\le k_i$, $\lim_{i\to+\infty}k_i'=+\infty$, and $\sigma(k_i')=\tau(k_i')$ $(i=1,2,\dots)$. Thus,

$$\sum_{j=\sigma(k_i)}^{k_i-1} p(j)\ge\sum_{j=\sigma(k_i)}^{k_i'-1} p(j)=\sum_{j=\sigma(k_i')}^{k_i'-1} p(j)=\sum_{j=\tau(k_i')}^{k_i'-1} p(j). \tag{2.32}$$

Combining inequalitites (2.31) and (2.32), we obtain

$$\liminf_{k\to+\infty}\sum_{i=\sigma(k)}^{k-1} p(i) \geq \liminf_{k\to+\infty}\sum_{i=\tau(k)}^{k-1} p(i) = \alpha. \tag{2.33}$$

The last inequality together with (2.30) imply (2.29).

Now assume, for the sake of contradiction, that u is a positive proper solution of Eq.(E). Then, for sufficiently large k, the function u is a positive proper solution of

$$\Delta u(k) + p(k)\, u(\sigma(k)) \leq 0.$$

By Lemma 2.1, the equation

$$\Delta u(k) + p(k)\, u(\sigma(k)) = 0 \tag{2.34}$$

has a positive proper solution $u_* : N_{n_0} \to (0,+\infty)$, where $n_0 \in N$ is sufficiently large.

Since (2.29) is satisfied, inequality (2.13) becomes

$$\limsup_{k\to+\infty}\frac{u_*(\sigma(k))}{u_*(k+1)} \leq \left(\frac{1+\sqrt{1-\alpha}}{\alpha}\right)^2 \tag{2.35}$$

or, if $p(k) \geq 1-\sqrt{1-\alpha}$ for sufficiently large k, then inequality (2.14) becomes

$$\limsup_{k\to+\infty}\frac{u_*(\sigma(k))}{u_*(k+1)} \leq \frac{1-\alpha+\sqrt{1-\alpha}}{\alpha^2}. \tag{2.36}$$

In the case that (2.35) holds, for any $\varepsilon \in (0, (1-\sqrt{1-\alpha})^2)$ and for sufficiently large k, we have

$$u_*(k+1) \geq \big((1-\sqrt{1-\alpha})^2 - \varepsilon\big) u_*(\sigma(k)). \tag{2.37}$$

Now, summing up Eq.(2.34) from $\sigma(k)$ to k, and using the fact that the function u_* is nonincreasing and the function σ is nondecreasing, we have

$$u_*(\sigma(k)) \geq u_*(k+1) + \left(\sum_{i=\sigma(k)}^{k} p(i)\right) u_*(\sigma(k)). \tag{2.38}$$

Combining inequalities (2.38) and (2.37), we obtain

$$u_*(\sigma(k)) \geq \left(\left(1-\sqrt{1-\alpha}\right)^2 - \varepsilon + \sum_{i=\sigma(k)}^{k} p(i)\right) u_*(\sigma(k)).$$

Hence

$$\limsup_{k\to+\infty} \sum_{i=\sigma(k)}^{k} p(i) \leq 1 - \left(1-\sqrt{1-\alpha}\right)^2 + \varepsilon,$$

which, for arbitrarily small values of ε, becomes

$$\limsup_{k\to+\infty} \sum_{i=\sigma(k)}^{k} p(i) \leq 1 - \left(1-\sqrt{1-\alpha}\right)^2.$$

This, contradicts (2.27).

In the case that (2.36) holds, following a similar procedure, we are led to the inequality

$$\limsup_{k\to+\infty} \sum_{i=\sigma(k)}^{k} p(i) \leq 1 - \alpha\frac{1-\sqrt{1-\alpha}}{\sqrt{1-\alpha}}$$

which contradicts (2.28). The proof is complete.

Remark 2.2 *If $\alpha > 1$, by (2.3), it is obvious, that the conditions of Theorem 2.1 are satisfied and therefore all proper solutions of Eq.(E) oscillate.*

Corollary 2.1 *Assume that*

$$0 < \alpha : \;= \liminf_{k\to+\infty} \sum_{i=k-n}^{k-1} p(i) \leq \left(\frac{n}{n+1}\right)^{n+1}$$

and

$$\limsup_{k\to+\infty} \sum_{i=k-n}^{k} p(i) > 1 - \left(1-\sqrt{1-\alpha}\right)^2. \tag{2.27$'$}$$

Then all proper solutions of Eq.(E_1) *oscillate.*

If, additionally, for sufficiently large k, $p(k) \geq 1-\sqrt{1-\alpha}$, and

$$\limsup_{k\to+\infty} \sum_{i=k-n}^{k} p(i) > 1 - \alpha\frac{1-\sqrt{1-\alpha}}{\sqrt{1-\alpha}}, \tag{2.28$'$}$$

then all proper solutions of Eq.(E_1) *oscillate.*

Now we present an example in which the condition (2.27′) of the above Corollary is satisfied, while none of the conditions (1.2), (1.3), (1.6), (1.7) and (1.9), is satisfied.

Example 1 *Consider the equation*

$$x(k+1) - x(k) + p(k)x(k-12) = 0, \quad k = 0, 1, 2, ...,$$

where

$$p(13k+1) = ... = p(13k+12) = \frac{35}{1200}, \; p(13k+13) = \frac{35}{1200} + \frac{6}{10}, \quad k = 0, 1, 2, ...$$

Here $n = 12$ *and it is easy to see that*

$$\alpha = \liminf_{k\to\infty} \sum_{i=k-12}^{k-1} p(i) = \frac{35}{100} < \left(\frac{12}{13}\right)^{13} \simeq 0.3532$$

$$\limsup_{k\to\infty} \sum_{i=k-12}^{k-1} p(i) = \frac{35}{100} + \frac{6}{10} = 0.950$$

and

$$\limsup_{k\to\infty} \sum_{i=k-12}^{k} p(i) = \frac{35}{1200} + \frac{950}{1000} = 0.9791 > 1 - \left(1 - \sqrt{1-\alpha}\right)^2 \simeq 0.9624.$$

We see that the condition (2.27') of Corollary 2.1 is satisfied and therefore all solutions oscillate. Observe, however, that

$$0.9791 < 1,$$

$$\alpha = 0.35 < \left(\frac{12}{13}\right)^{13} \simeq 0.3532,$$

$$0.950 < 1 - \frac{\alpha^2}{4} \simeq 0.9693,$$

$$0.950 < 1 - \alpha^{12} \simeq 0.9999,$$

and

$$0.950 < 1 - \frac{\alpha^2}{2(2-\alpha)} \simeq 0.9628.$$

Therefore none of the conditions (1.2), (1.3), (1.6), (1.7) and (1.9), is satisfied.

Theorem 2.3 *Assume that* $\alpha \in (0,1]$ *and there exist* $n_0 \in N$ *and a function* $\widetilde{p} \in L_{\mathrm{loc}}(R_+, R_+)$ *such that*

$$t^2\,\widetilde{p}(t) \text{ is nondecreasing function, } \widetilde{p}(i) \le p(i) \text{ for } i \in N_{n_0} \tag{2.39}$$

and

$$\liminf_{k\to+\infty} \int_{\tau(k)-1}^{k-1} \widetilde{p}(s)ds \ge \alpha. \tag{2.40}$$

Then condition (2.27) (*or, if for sufficiently large* k, $p(k) \ge 1 - \sqrt{1-\alpha}$, *condition* (2.28)) *is sufficient for all proper solutions of Eq.*(E) *to oscillate.*

Proof. In view of Lemma 2.1 and Theorem 2.2, to prove Theorem 2.3, it sufficies to show that

$$\liminf_{k\to+\infty} \sum_{i=\tau(k)}^{k-1} p(i) \ge \alpha. \tag{2.41}$$

By (2.39) and (2.40), we have

$$\sum_{i=\tau(k)}^{k-1} p(i) \ge \sum_{i=\tau(k)}^{k-1} \frac{(i-1)i^2}{i}\widetilde{p}(i)\int_{i-1}^{i}\frac{ds}{s^2} \ge \sum_{i=\tau(k)}^{k-1}\frac{i-1}{i}\int_{i-1}^{i}\widetilde{p}(s)\,ds \ge$$
$$\ge \frac{\tau(k)-1}{\tau(k)}\sum_{i=\tau(k)}^{k-1}\int_{i-1}^{i}\widetilde{p}(s)\,ds = \frac{\tau(k)-1}{\tau(k)}\int_{\tau(k)-1}^{k-1}\widetilde{p}(s)\,ds. \tag{2.42}$$

Since $\tau(k) \to +\infty$ for $k \to +\infty$, inequality (2.42), in view of (2.40), implies (2.41). The proof is complete.

Corollary 2.2 *Consider Eq.(E) and let* $c \in (0,+\infty)$, $\beta \in (0,1)$, $c\ln\beta \ge -1$ *and for large* k

$$p(k) \ge \frac{c}{k}, \quad \tau(k) \le [\beta k],$$

and

$$\limsup_{k\to+\infty} \sum_{i=[\beta k]}^{k} p(i) > 1 - \left(1-\sqrt{1-\alpha}\right)^2,$$

where $\alpha = \ln\beta^{-c}$ *and* $[\beta k]$ *denotes the integer part of* βk. *Then all proper solutions of Eq.*(E) *oscillate.*

Proof. Take $\widetilde{p}(t) = \frac{c}{t}$ and $\alpha = \ln \beta^{-c}$. Then it is easily shown that the conditions of Theorem 2.3 are satisfied.

Analogously, if we take $\widetilde{p}(t) = \frac{c}{t \ln t}$, we have the following

Corollary 2.3 *Consider Eq.(E) and let* $c \in (0, +\infty)$, $\beta \in (0, 1)$, $c \ln \beta \geq -1$ *and for large* k

$$p(k) \geq \frac{c}{k \ln k}, \quad \tau(k) \leq [k^{\beta}].$$

and

$$\limsup_{k \to +\infty} \sum_{i=[k^{\beta}]}^{k} p(i) > 1 - \left(1 - \sqrt{1-\alpha}\right)^2,$$

where $\alpha = \ln \beta^{-c}$ *and* $[k^{\beta}]$ *denotes the integer part of* k^{β}. *Then all proper solutions of* Eq.(E) *oscillate.*

References

[1] R. P. Agarwal and P. J. Y. Wong, Advanced topics in difference equations. *Kluwer Academic Publishers*, 1997

[2] G. E. Chatzarakis and I. P. Stavroulakis, Oscillations of First Order Linear Delay Difference Equations. *The Australian J. Math. Analysis and Appl.*, 2006 (in press).

[3] S. S. Cheng and G. Zhang, "Virus" in several discrete oscillation theorems. *Appl. Math. Lett.* **13** (2000), No. 3, 9–13.

[4] Y. Domshlak, A discrete analogue of the Sturm comparison theorem for non-symmetric equations. (Russian) *Dokl. Akad. Nauk Az. SSR* **37** (1981), No. 10, 12–15.

[5] Y. Domshlak, What should be a discrete version of the Chanturia-Koplatadze lemma? *Funct. Differ. Equ.* **6** (1999), No. 3-4, 299–304.

[6] L. Erbe and B. G. Zhang, Oscillation of discrete analogues of delay equations. *Differ. Integral Equ.* **2** (1989), No. 3, 300–309.

[7] I. Győri and G. Ladas, Oscillation theory of delay differential equations with applications. *Clarendon Press. Oxford*, 1991.

[8] R. G. Koplatadze and T. A. Chanturia, On oscillatory and monotone solutions of first order differential equations with retarded argument. (Russian) *Differentsial'nye Uravneniya* **18** (1982), No. 8, 1463–1465.

[9] G. Ladas, C. Philos and Y. Sficas, Sharp conditions for the oscillation of delay difference equations. *J. Appl. Math. Simulation* **2** (1989), No. 2, 101–112.

[10] B. Lalli and B. G. Zhang, Oscillation of difference equations. *Colloquium Math.* **65** (1993), No. 1, 25–32.

[11] Ch. G. Philos, I.K. Pournaras and I. P. Stavroulakis, Sufficient conditions for the oscillation of delay difference equations, *J. Difference Equ. Appl.*, **10** (2004), 419-435.

[12] Y. G. Sficas and I. P. Stavroulakis, Oscillation criteria for first-order delay equations. *Bull. Lond. Math. Soc.* **35** (2003), No. 2, 239–246.

[13] J. H. Shen and I. P. Stavroulakis, Oscillation criteria for delay difference equations. *Electron. J. Differ. Equ.* **2001** (2001), No. 10, 1–15.

[14] I. P. Stavroulakis, Oscillations of delay difference equations. *Comput. Math. Appl.* **29** (1995), No. 7, 83–88.

[15] I. P. Stavroulakis, Oscillation criteria for first order delay difference equations. *Mediterr. J. Math.* **1** (2004), No. 2, 231–240.

[16] J. S. Yu, B. G. Zhang and X. Z. Qian, Oscillations of delay difference equations with oscillating coefficients. *J. Math. Anal. Appl.* **177** (1993), No. 2, 432–444.